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EFFICIENT ITERATIVE ARMA APPROXIMATION OF MULTIVARIATE RANDOM PROCESSES FOR STRUCTURAL DYNAMICS APPLICATIONS

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SUMMARY

An efficient Auto-Regressive Moving–Average (ARMA) approximation method is presented for simulating stationary random processes with specified (target) power spectra in conjunction with structural dynamics applications. It involves an iterative algorithm developed for minimizing a physically motivated 'energy' measure, in the frequency domain, of the ARMA approximation of an AR representation of the target spectrum. The iterative algorithm can be used to adjust, for better spectral matching, the parameters of an arbitrary ARMA approximation of the random process determined by any other method; this is accomplished without increasing the requisite order of the ARMA approximation. The efficiency of the proposed method is demonstrated by considering spectra which are commonly used in earthquake engineering and ocean engineering.

KEY WORDS: random processes; efficient simulation, ARMA algorithms, earthquakes; ocean waves

1. INTRODUCTION

Over a period of years considerable effort has been devoted to the problem of simulating stochastic processes with specified spectral characteristics; synthesized time histories can be readily used for dynamic analysis of complex engineering structures. The Auto-Regressive Moving-Average (ARMA) method has been established as an efficient tool for this purpose. It requires constructing an ARMA filter which adequately approximates the target spectrum. A two-stage ARMA approximation is one of the most versatile techniques in this regard;[1–3] a 'long' Auto-Regressive (AR) filter is constructed first and, then, the AR filter is approximated by a 'short' ARMA filter.

This paper presents an efficient method for the two-stage ARMA approximation. An error criterion which quantifies the 'energy' of the discrepancy between the AR and ARMA approximations is utilized. An itrerative algorithm is developed to solve the associated error minimization problem. In this context the method can also be used to adjust the parameters of an arbitrary ARMA approximation of a random process, towards achieving improved spectral matching. To indicate the robustness of the proposed approximation method, spectra that are commonly used in offshore and earthquake engineering are considered.

2. AR APPROXIMATION

In this section the properties of an AR process[4,5] are reviewed first, since the proposed ARMA simulation method requires that the target process is initially approximated by an AR model.

The value at time k of an n-variate AR process $\underline{y}$ of order m can be computed as

$$\underline{y}(k) = -\sum_{i=1}^{m} a_i^{\mathrm{AR}} \underline{y}(k-i) + b_0^{\mathrm{AR}} \underline{w}(k) \tag{1}$$

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where $\underline{w}$ is an n-variate band-limited $[-\omega_b, \omega_b]$ white noise process

$$E[\underline{w}(k)(\underline{w}(l))^t] = 2\omega_b I_n \delta_{kl}. \tag{2}$$

The symbols $E[\ ]$ and $(\ )^t$ denote the operators of mathematical expectation and transposition, respectively; I_n is the unit matrix, and δ_{kl} is the Kronecker delta. The sampling period and the cut-off frequency are related by the well-known Nyquist relation

$$T = \pi/\omega_b. \tag{3}$$

Alternatively, the process defined by equation (1) can be deemed as the white noise response of a discrete linear system with the following transfer function

$$H_{AR}(z) = A_{AR}^{-1}(z) b_0^{AR} \tag{4}$$

where

$$A_{AR}(z) = \sum_{k=0}^{m} a_k^{AR} z^{-k} \tag{5}$$

$z = e^{j\omega T}$ and $a_0 = I_n$. The spectral matrix of the AR process can be found from the equation

$$S_{AR}(\omega) = H_{AR}^*(z) H_{AR}^t(z) \tag{6}$$

where $(\ )^*$ denotes complex conjugation.

The AR parameters can be evaluated by multiplying equation (1) by $y^t(k-l)$ and taking mathematical expectation. As the future input excitation is statistically independent from the present system response, the following set of Yule–Walker equations can be derived:

$$R_{\underline{y}}^t(k) + \sum_{i=1}^{m} a_i^{AR} R_{\underline{y}}(i-k) = 0, \quad k = 1, \ldots, m \tag{7}$$

where the autocorrelation function $R_{\underline{y}}(k)$ is equated to the autocorrelation function of the target process at the first $m+1$ time lags. If the target spectrum is specified, the target auto-correlation function can be determined by relying on the equation

$$R_{\underline{y}}(k) = \int_{-\omega_b}^{\omega_b} S_{\underline{y}}(\omega) \cos(k\omega T)\, d\omega \tag{8}$$

The gain factor b_0^{AR} of the AR model can be estimated by using the following equation:

$$b_0^{AR}(b_0^{AR})^t = \frac{1}{2\omega_b}\left[R_{\underline{y}}(0) + \sum_{i=1}^{m} a_i^{AR} R_{\underline{y}}(i)\right] \tag{9}$$

Note that the solution of equation (9) is not unique. In this study the matrix b_0^{AR} is determined by performing the Cholevsky factorization of the expression on the right-hand side of equation (9). Upon evaluating the AR parameters, some important characteristics of the AR process can be determined. Specifically, the input–output cross-correlation $R_{\underline{y}\underline{w}}(l) = E[\underline{y}(k)\underline{w}(k+l)]$ can be evaluated recursively by using the equations

$$R_{\underline{y}\underline{w}}(l) = 0, \quad l > 0 \tag{10}$$

$$R_{\underline{y}\underline{w}}(l) = 2\omega_b b_0^{AR} \quad l = 0 \tag{11}$$

$$R_{\underline{y}\underline{w}}(l) = -\sum_{i=1}^{\min(m, -l)} a_i^{AR} R_{\underline{y}\underline{w}}(l+i) \quad l < 0 \tag{12}$$

Also, the autocorrelation function of the AR process satisfies the following equation:

$$R_{\underline{y}}(-k) + \sum_{i=1}^{m} a_i^{\mathrm{AR}} R_{\underline{y}}(i-k) = b_0^{\mathrm{AR}} R_{\underline{yw}}(-k), \quad k = 0, \pm 1, \ldots \tag{13}$$

Note that Yule–Walker equations can be derived from equation (13) by restricting k to the values $1, 2, \ldots, m$.

3. NEW ARMA APPROXIMATION

3.1. *Definition of ARMA system*

The kth sample of an n-variate ARMA(p, q) process $\underline{y}$ can be generated using the equation

$$\underline{y}(k) = -\sum_{i=1}^{p} a_i \underline{y}(k-i) + \sum_{l=0}^{q} b_l \underline{w}(k-l) \tag{14}$$

The ARMA process can also be interpreted as the white noise response of a discrete linear system with the transfer function

$$H_{\mathrm{ARMA}}(z) = A^{-1}(z) B(z) \tag{15}$$

where

$$A(z) = \sum_{i=0}^{p} a_i z^{-i}, \quad a_0 = I_n \tag{16}$$

and

$$B(z) = \sum_{l=0}^{q} b_l z^{-1}. \tag{17}$$

3.2. *Review of methods for ARMA parameter evaluation*

Several alternative procedures to obtain unknown ARMA coefficients are utilized in practice. The commonly used Auto-Cross-Correlation Matching (ACM) method has been developed by Gersch and Yonemoto[1] and consequently has been applied to engineering mechanics problems in References 2 and 3. Multiplying equation (14) by $\underline{y}^{\mathrm{t}}(k-j)$ and $\underline{w}^{\mathrm{t}}(k-j)$ and taking mathematical expectation yields a system of linear equations with unknown ARMA parameters:

$$\sum_{i=0}^{p} a_i R_{\underline{yy}}(i-k) - \sum_{l=0}^{q} b_l R_{\underline{wy}}(l-k) = 0, \quad k = 1, \ldots, p \tag{18}$$

$$-\sum_{i=0}^{p} a_i R_{\underline{yw}}(i-k) + 2\omega_{\mathrm{b}} b_k = 0, \quad k = 0, \ldots, q \tag{19}$$

However, these equations contain additional unknown input–output cross-correlation function $R_{\underline{yw}}(k)$. The ACM method assumes that the auto-correlation function of the ARMA process is equal to the target auto-correlation function at the first $p + 1$ time legs, and the input–output cross-correlation of the ARMA process is found by using the predetermined AR model based on equations (10)–(12). Mignolet and Spanos[6,7] have shown that this method is equivalent to the minimization of the error criterion

$$\varepsilon_1 = \frac{1}{2\omega_{\mathrm{b}}} \int_{-\omega_{\mathrm{b}}}^{\omega_{\mathrm{b}}} |A(z) H_{\mathrm{AR}}(z) - B(z)|^2 \, \mathrm{d}\omega \tag{20}$$

with the constrain $a_0 = I_n$. The symbol $|U|$ denotes the Euclidean norm of an arbitrary matrix U. That is,

$$|U|^2 = \mathrm{tr}(U^* U^{\mathrm{t}}) = \mathrm{tr}((U^{\mathrm{t}})^* U) \tag{21}$$

where tr( ) stands for the trace operator.

Alternatively, the variance of the difference between the responses to the white noise of the ARMA and AR systems can be minimized:

$$\varepsilon_2 = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} |H_{AR}(z) - A^{-1}(z)B(z)|^2 \, d\omega = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} |H_{AR}(z) - H_{ARMA}(z)|^2 \, d\omega \tag{22}$$

This error lends itself to immediate physical interpretation. From a deterministic perspective, the error criterion (22) represents the 'energy' of the difference between the impulse responses of the AR and ARMA systems. That is, this error criterion can be experessed in the time domain as

$$\varepsilon_2 = \sum_{k=0}^{\infty} |h_{AR}(k) - h_{ARMA}(k)|^2 \tag{23}$$

where h_{AR} and h_{ARMA} are the impulse responses of the AR and ARMA systems, respectively. Clearly, the error criterion defined by equations (22) and (23) has a readily discernible physical sense and is more appealing for engineering approximations. Note that equation (20) can be equivalently rewritten as

$$\varepsilon_1 = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} |A(z)(H_{AR}(z) - H_{ARMA}(z))|^2 \, d\omega \tag{24}$$

That is, the error criterion ε_1 is the integral of the difference between the transfer functions of the AR and ARMA systems weighted by the matrix $A(e^{j\omega T})$; this matrix tends to decrease the influence of $H_{AR}(z) - H_{ARMA}(z)$ on the frequencies where the magnitude of the transfer function $|H_{ARMA}(z)| = |A^{-1}(z)B(z)|$ is large. In particular, the ACM method provides a less-accurate approximation for systems with poles close to the unit circle.

Note that an 'energy' based reduction of long AR systems to short ARMA models has been previously pursued in Reference 8 by identifying the modes which are associated with the largest amount of energy. This procedure has also been examined for ocean engineering problems by Mandal *et al.*,[9] and for aerospace engineering applications by Spanos *et al.*[10]

Clearly, the error criterion ε_2 is not a quadratic function of the unknown ARMA parameters and, in general, requires application of non-linear optimization schemes. In this regard, Porat and Friedlander,[11] and Spanos *et al.*[12] have utilized a version of the steepest descent optimization algorithm by establishing analytical expressions for the corresponding gradients. Cheng and Chang[13] have reduced this optimization problem to the linear programming for the model reduction application.

In this paper an iterative algorithm is adopted for the minimization of the criterion ε_2. Similar iterations were used in References 14 and 15 for system identification applications. At every iteration the ARMA parameters which are determined from the previous iteration are used to improve the spectral estimate.

3.3 New method formulation: univariate case

In the univariate case one can introduce the sequence

$$\varepsilon_i = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} \left|\frac{A_i(z)}{A_{i-1}(z)}\right|^2 \left|H_{AR}(z) - \frac{B_i(z)}{A_i(z)}\right|^2 d\omega = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} |A_i(z)Q_1^{i-1}(z) - B_i(z)Q_2^{i-1}(z)|^2 \, d\omega \tag{25}$$

where

$$Q_1^{i-1}(z) = \frac{H_{AR}(z)}{A_{i-1}(z)}, \qquad Q_2^{i-1}(z) = \frac{1}{A_{i-1}(z)} \tag{26}$$

Note that if $A_{i-1}(z) = 1$, equation (25) reduces to equation (20). However, when $A_i(z) = A_{i-1}(z)$, equation (25) represents the 'energy' error criterion (22). Thus, the following condition for convergence of the algorithm is specified:

$$|\varepsilon_2^i - \varepsilon_2^{i-1}|/\varepsilon_2^i < \delta \tag{27}$$

where

$$\varepsilon_2^i = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} |H_{AR}(z) - B_i(z)/A_i(z)|^2 \, d\omega \tag{28}$$

and δ is a number reflecting acceptable error.

After performing some algebraic manipulations, equation (25) can be rewritten in an equivalent form:

$$\varepsilon_i = \frac{1}{2\omega_n}\left(\sum_{k=0}^{p}\sum_{l=0}^{p} a_k R_{11}(k-l)a_l - \sum_{k=0}^{q}\sum_{l=0}^{p} b_k R_{12}(l-k)a_l \right.$$
$$\left. - \sum_{k=0}^{p}\sum_{l=0}^{q} a_k R_{12}(k-l)b_l - \sum_{k=0}^{q}\sum_{l=0}^{q} b_k R_{22}(k-l)b_l \right) \tag{29}$$

where the superscript index indicating the number of iterations is omitted for notational convenience, and

$$R_{rl}(k) = \int_{-\omega_b}^{\omega_b} Q_r^*(z)Q_l(z)e^{jk\omega T} \, d\omega, \quad r, l = 1, 2 \tag{30}$$

Clearly, R_{11} and R_{22} can be thought of as the auto-correlation functions of AR processes y_1 and y_2 with the transfer functions $Q_1(z)$ and $Q_2(z)$ given by equation (26). Similarly, R_{12} can be viewed as the cross-correlation of y_1 and y_2. These auto- and cross-correlation functions can be found by constructing the state-space realization[16] of these systems and solving the corresponding Lyapunov equations as it is shown in the appendix.

The error criterion ε_i expressed by equation (29) is a positive-definite quadratic form involving the unknown ARMA parameters. For its minimization it is sufficient to set the derivatives of ε_i with respect to the ARMA parameters a_k^i, $k = 1, \ldots, p$, and b_l^i, $l = 0, \ldots, q$ equal to zero. This leads to a small dimension system of linear equations involving a symmetric positive-definite matrix. Specifically,

$$\sum_{i=0}^{p} a_i R_{11}(i-k) - \sum_{l=0}^{q} b_l R_{21}(l-k) = 0, \quad k = 1, \ldots, p \tag{31}$$

$$-\sum_{i=0}^{p} a_i R_{12}(i-k) + \sum_{l=0}^{q} b_l R_{22}(l-k) = 0, \quad k = 0, \ldots, q \tag{32}$$

where $a_0 = 1$.

Note that if the iterative algorithm is initiated with $A^0(z) = 1$, the proposed method yields the ACM solution as the first iteration. Further iterations are performed until the condition specified by equation (27) is satisfied.

3.4. New method formulation: multivariate case

The proposed method can be generalized for the multivariate case by introducing the sequence

$$\varepsilon_i = \frac{1}{2\omega_b}\int_{-\omega_b}^{\omega_b} |A_i(z)(A_{i-1}^{-1}(z)H_{AR}(z)) - A_{i-1}^{-1}(z)B_i(z)|^2 \, d\omega \tag{33}$$

Again, if $A_{i-1}(z) = I_n$, equation (33) reduces to equation (20) which provides the ACM solution. However, when $A_i(z) = A_{i-1}(z)$, equation (33) represents the error criterion (22). Then the convergence of this iterative procedure is determined by equation (27) where ε_2^i is given by equation (28). However, since the general matrices do not commute, the straightforward minimization of equation (33) leads to algebraic expressions which are not amenable to immediate physical interpretation. To circumvent this problem, the procedure of minimizing the error criterion (33) is pursued in two steps.

During the first stage, ε_2^i is only minimized with respect to the AR parameters a_k^i, $k = 1, \ldots, p$, for the specified MA parameters b_l, $l = 0, \ldots, q$. Performing similar manipulations as in the univariate case, ε_i can be rewritten in the equivalent form

$$\varepsilon_i = \frac{1}{2\omega_b} \mathrm{tr}\left(\sum_{k=0}^{p} \sum_{l=0}^{p} a_k R_{11}(k-l) a_l^t - \sum_{l=0}^{p} R_{21}(-l) a_l^t - \sum_{l=0}^{p} a_l R_{12}(l) + R_{22}(0) \right) \tag{34}$$

where

$$R_{rl}(k) = \int_{-\omega_b}^{\omega_b} Q_r^*(z) Q_l^t(z) e^{jk\omega T} \, d\omega, \quad r, l = 1, 2 \tag{35}$$

is the auto/cross-correlation of the processes with the transfer functions

$$Q_1^{i-1}(z) = A_{i-1}^{-1}(z) H_{AR}(z) = (A_{AR}(z) A_{i-1}(z))^{-1} b_0^{AR} \tag{36}$$

and

$$Q_2^{i-1}(z) = A_{i-1}^{-1}(z) B_{i-1}(z) \tag{37}$$

An efficient procedure to evaluate the auto- and cross-correlation functions defined by equation (35) is given in the appendix.

Again, the minimum of ε_i can be obtained by setting its derivatives with respect to a_k, $k = 1, \ldots, p$, equal to zero. This leads to the following system of linear equations:

$$\sum_{i=0}^{p} a_i R_{11}(i-k) - R_{21}(-k) = 0, \quad k = 1, \ldots, p \tag{38}$$

where $a_0 = I_n$. The system of equation (38) can be readily solved, yielding a new estimate of the AR parameters.

Similarly, during the second stage the optimal error criterion $\varepsilon_i = \varepsilon_2$ is only minimized with respect to the MA parameters b_l^i, $l = 0, \ldots, p$, for the given AR part a_k, $k = 0, \ldots, q$. The associated error criterion can be expressed as

$$\varepsilon_2 = \frac{1}{2\omega_b} \mathrm{tr}(R_{11}(0) - \sum_{i=0}^{q} b_i^t R_{21}(i) - \sum_{i=0}^{q} R_{12}(-i) b_i + \sum_{k=0}^{q} \sum_{i=0}^{q} b_i^t R_{22}(k-i) b_k) \tag{39}$$

where the cross- and auto-correlation functions which appear in equation (39) are defined by equation (35) with

$$Q_1(z) = H_{AR}^t(z) = (b_0^{AR})^t (A_{AR}^t(z))^{-1} = ((b_0^{AR})^t A_{AR}^t(z) (b_0^{AR})^{-t})^{-1} (b_0^{AR})^t \tag{40}$$

and

$$Q_2^{i-1}(z) = A_i^{-t}(z) \tag{41}$$

Setting the derivatives of ε_i with respecrt to b_k^t, $k = 0, \ldots, p$, equal to zero yields the system of linear equations

$$\sum_{i=0}^{q} b_i^t R_{22}(i-k) - R_{12}(-k) = 0, \quad k = 0, \ldots, q \tag{42}$$

This system of equations can be readily solved to determine a new estimate for the MA parameters. Then, the convergence criterion (27) can be evaluated and a new iteration cycle can be performed, if necessary.

Interestingly, a similar iterative ARMA modelling method has independently been proposed by Friot and Bouc.[17] However, the authors consider this development as a way to improve the ACM method rather than a new 'energy' based approximation of the target spectra. Also, they consider only scalar stochastic processes, whereas the present paper deals with the multi-variate stochastic processes. Finally, this paper recognizes that

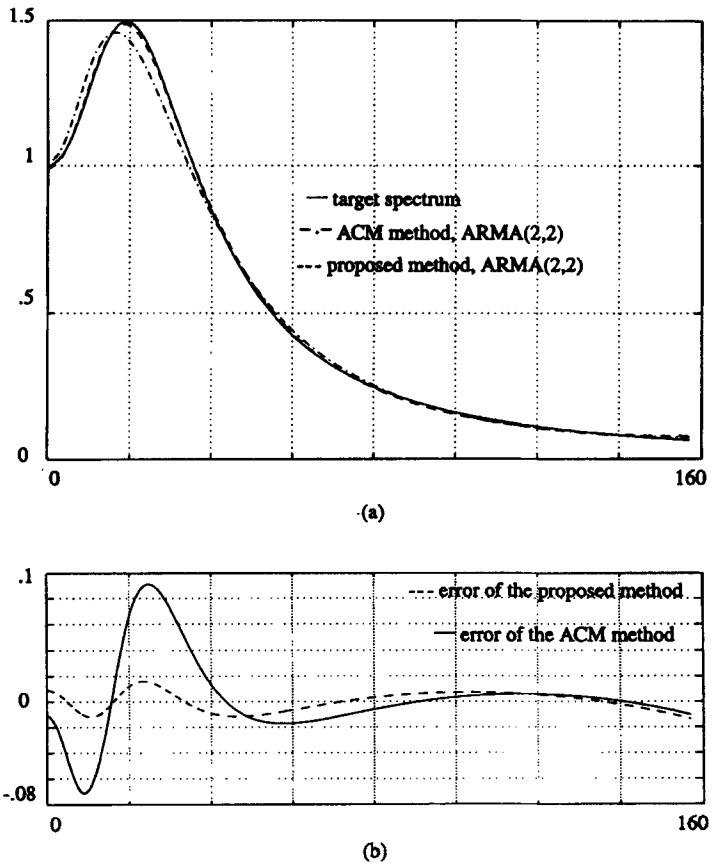

Figure 1. (a) Target Kanai–Tajimi spectrum and its approximations; (b) error of approximations

the integrals in equations (30) and (35) signify the auto-correlation function of two known ARMA systems and can be found by solving pertinent Lyapunov equations; see the appendix. Friot and Bouc[17] utilize the less computationally attractive FFT algorithm to evaluate numerically these integrals.

4. NUMERICAL EXAMPLES

The iterative procedure developed in the previous section has been applied to spectra which are used in offshore and earthquake engineering.

The Kanai–Tajimi spectrum[18]

$$S(\omega) = S_0 \frac{\omega_g^4 + 4\zeta_g^2 \omega_g^2 \omega^2}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2 \omega_g^2 \omega^2} \tag{43}$$

is selected for the first application; the values of ζ_g, ω_g, and S_0 are 8π, 0·8, and 1, respectively. The results of ARMA(2,2) modelling are shown in Figure 1(a). The approximation errors of the ACM method and of the proposed method are shown in Figure 1(b). Clearly, the proposed method can yield improved spectral matching.

The Pierson–Moskowitz spectrum[19]

$$S(\omega) = \frac{e^{5/4}}{\omega^5} \exp\left(-\frac{5}{4\omega^4}\right) \tag{44}$$

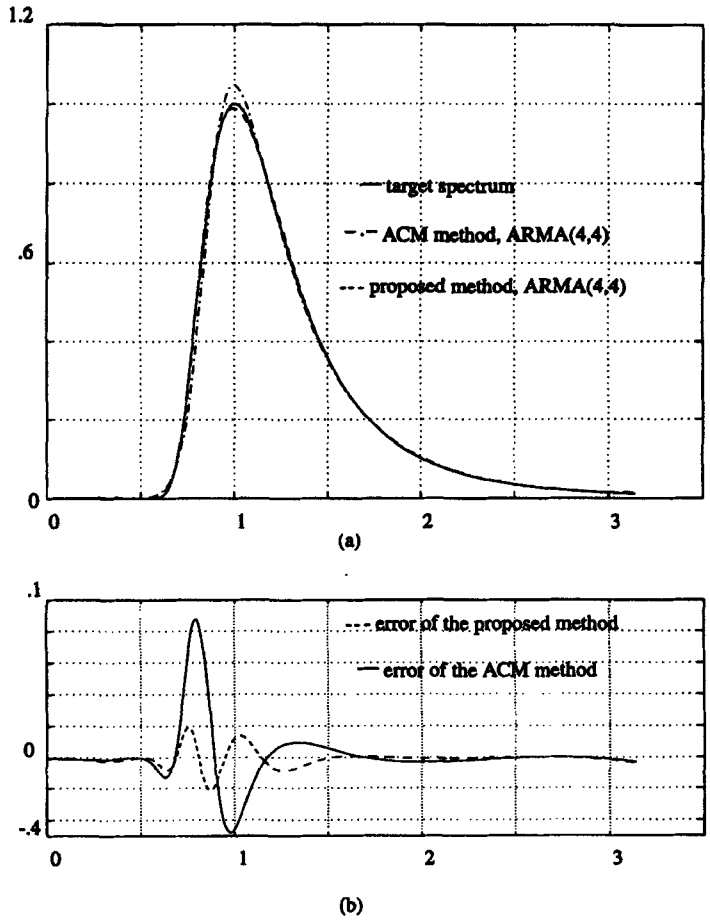

Figure 2. (a) Target Pierson–Moskowitz spectrum and its approximations; (b) error of approximations

is often used in ocean engineering. ARMA(4,4) approximations derived by the proposed method and the ACM method, along with corresponding approximation errors, are given in Figures 2(a) and 2(b). It is seen again that the proposed iterative method leads to improved spectral representation.

Finally, a two-variate random process which describes the seismic motion of two ground sites is approximated by using the proposed ARMA modelling method. This spatial correlation model has been proposed by Harichandran and Vanmarcke[18] for treating the data recorded during the Taiwan earthquake of 17 December 1982. The spectral form is

$$S(\omega) = \begin{bmatrix} S_{11}(\omega) & \rho_{12}(\omega)\sqrt{S_{11}(\omega)S_{22}(\omega)} \\ \rho_{12}(\omega)\sqrt{S_{11}(\omega)S_{22}(\omega)} & S_{22}(\omega) \end{bmatrix} \tag{45}$$

where $S_{11}(\omega) = S_{22}(\omega)$ is assumed to be the Kanai–Tajimi spectrum given by equation (43); the same numerical values for the coefficients are used. The cross-correlation function is[18]

$$\rho_{12}(\omega) = 0{\cdot}736\exp\left(-\frac{5{\cdot}063r}{\theta(\omega)}\right) + 0{\cdot}294\exp\left(-\frac{0{\cdot}744r}{\theta(\omega)}\right) \tag{46}$$

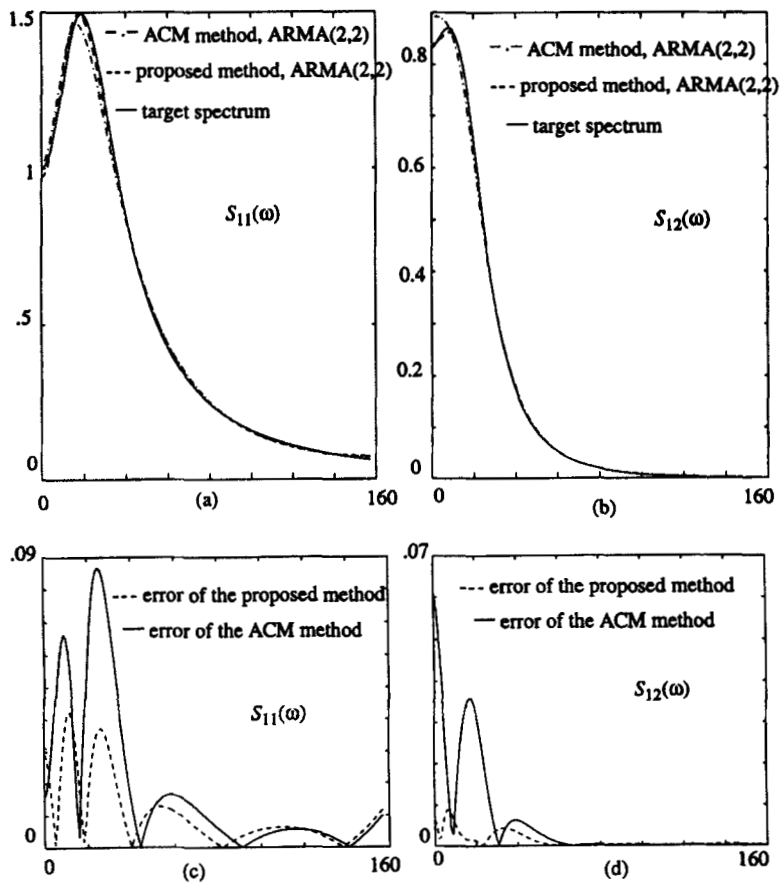

Figure 3. (a) The $S_{11}(\omega)$ component of the target two-variate spectrum and its approximations, (b) the $S_{12}(\omega)$ component of the target two-variate spectrum and its approximations, (c) error of approximations of $S_{11}(\omega)$, (d) error of approximations of $S_{12}(\omega)$

where

$$\theta(\omega) = 5{\cdot}210\sqrt{1 + \left(\frac{\omega}{2{\cdot}18\pi}\right)^{2{\cdot}78}} \tag{47}$$

and r denotes the distance between two sites. The constant r is selected equal to 0·3. Note that this multivariate spectrum is not a rational function of frequency. ARMA(2, 2) approximations which are determined by using the proposed method and the ACM method are given in Figure 3. Clearly, the 'energy' error criterion leads to a closer approximation of the target spectrum, especially for the range of frequencies associated with the spectral peak, without increasing the requisite order of the ARMA approximation.

Note that for the considered numerical examples two iterations have been found sufficient to reach convergence in the sense of equation (27).

5. CONCLUDING REMARKS

An efficient method of ARMA spectral approximation, in conjunction with the problem of simulating multivariate random processes, has been developed by adopting a frequency domain error criterion which lends itself, readily, to physical interpretation. The method is iterative and requires solving linear equations only; this can be efficiently performed by modern digital computers. Pertinent numerical examples have shown the usefulness of the method for practical applications; it yields low-order, and thus computationally

efficient, ARMA simulation algorithms even for spectra, such as the Pierson–Moskowitz spectrum of ocean waves, which are mathematically intractable for AR approximation. It is noted that the proposed method exhibits the desirable features of minimizing a physically discernible spectral matching error, and of requiring the solution of linear equations only. Further, it is versatile in the sense that it can be used to adjust, for better spectral matching, the parameters of an arbitrary ARMA process determined by any other method. For this purpose, the ARMA approximation, which is to be improved, merely needs to be introduced as the initial iterant in equation (25) or equation (33); improved spectral matching will be achieved without increasing the order of the ARMA approximation.

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APPENDIX

Determination of the cross-correlation function of ARMA processes

Consider two ARMA processes, $\underline{y}_1(k)$ and $\underline{y}_2(k)$, which are specified by the transfer functions

$$Q_1(z) = A_1^{-1}(z)B_1(z) \quad \text{and} \quad Q_2(z) = A_2^{-1}(z)B_2(z) \tag{48}$$

where

$$A_r(z) = \sum_{k=0}^{m} a_k^r z^{-k}, \quad B_r(z) = \sum_{k=0}^{m} b_k^r z^{-k}, \qquad r = 1, 2 \tag{49}$$

and $a_0^r = I_n$. The cross-correlation function of these random processes can be evaluated by solving a system of linear equations. This system of equations can be assembled by multiplying the input–output relationship of the ARMA model (14) by the pertinent input and output quantities with subsequent averaging. However, the following procedure is more efficient for numerical calculations and it has been used in the numerical results of the present study.

The transfer function of the ARMA system can be written using different order of multiplication of its AR and MA parts That is,

$$Q_r(z) = A_r^{-1}(z)B_r(z) = \tilde{B}_r(z)\tilde{A}_r^{-1}(z), \quad r = 1, 2 \tag{50}$$

where the parameters $\tilde{a}_k^r$, 1, ... ,p_r, and $\tilde{b}_k^r$, 0, ... q_r can be found by solving the following equation:

$$A_r(z)\tilde{B}_r(z) = B_r(z)\tilde{A}_r(z), \quad r = 1, 2 \tag{51}$$

Then, the cross-correlation $R_{12}(k)$ of the ARMA processes $\underline{y}_1(k)$ and $\underline{y}_2(k)$ can be determined by using the equation

$$\begin{aligned} R_{12}(k) &= \int_{-\omega_b}^{\omega_b} Q_1^*(z)Q_2^t(z)z^k \, d\omega = \sum_{m=0}^{q_1}\sum_{l=0}^{q_2} \tilde{b}_m^1 \left[\int_{-\omega_b}^{\omega_b} \tilde{A}_1^{*-1}(z)\tilde{A}_2^{-t}(z)z^{k-l+n} \, d\omega\right] \tilde{b}_l^2 \\ &= \sum_{m=0}^{q_1}\sum_{l=0}^{q_2} \tilde{b}_m^1 \tilde{R}_{12}(k-l+n)\tilde{b}_l^2 \end{aligned} \tag{52}$$

where $\tilde{R}_{12}(k)$ is the cross-correlation function of two AR processes $\underline{\tilde{y}}_1(k)$ and $\underline{\tilde{y}}_2(k)$ with the following transfer functions:

$$Q_1(z) = \tilde{A}_1^{-1}(z) \quad \text{and} \quad Q_2(z) = \tilde{A}_2^{-1}(z) \tag{53}$$

respectively. This cross-correlation can be found by using the equivalent state-space representation for the AR processes[16]

$$v_r(k) = F_r v_r(k-1) + C_r w(k), \quad r = 1, 2, \tag{54}$$

where

$$v_r(k) = \begin{bmatrix} \tilde{y}_r(k-p_r+1) \\ \tilde{y}_r(k-p_r+2) \\ \cdots \\ \tilde{y}_r(k) \end{bmatrix}, \quad F_r = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & I_n & \cdots & 0_n \\ 0_n & 0_n & 0_n & \cdots & 0_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\tilde{a}^r_{p_r} & -\tilde{a}^r_{p_r-1} & -\tilde{a}^r_{p_r-2} & \cdots & -\tilde{a}^r_1 \end{bmatrix}, \quad C_r = \begin{bmatrix} 0_n \\ 0_n \\ \cdots \\ I_n \end{bmatrix} \tag{55}$$

Multiplying equation (54) by its transposition and taking mathematical expectation one can derive the following Lyapunov equation:

$$\tilde{R}_{12} = F_1 \tilde{R}_{12} F_2^t + 2\omega_b C_1 C_2^t \tag{56}$$

where

$$\tilde{R}_{12} = \begin{bmatrix} \tilde{R}_{12}(p_1 - p_2) & \cdots & \tilde{R}_{12}(p_1 - 1) \\ \cdots & \cdots & \cdots \\ \tilde{R}_{12}(1 - p_2) & \cdots & \tilde{R}_{12}(0) \end{bmatrix} \tag{57}$$

is the Toeplitz matrix which contains the elements of the cross-correlation function of the AR systems.

Thus, calculation of the correlation function of two ARMA systems can be performed based on equations (52) and (55).

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